Exact retrospective Monte Carlo computation of arithmetic average Asian options

Benjamin Jourdain, Mohamed Sbai

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Benjamin Jourdain[1] and *Mohamed Sbai*[1]

Abstract

Taking advantage of the recent literature on exact simulation algorithms (Beskos *et al.* [1]) and unbiased estimation of the expectation of certain functional integrals (Wagner [23], Beskos *et al.* [2] and Fearnhead *et al.* [6]), we apply an exact simulation based technique for pricing continuous arithmetic average Asian options in the Black & Scholes framework. Unlike existing Monte Carlo methods, we are no longer prone to the discretization bias resulting from the approximation of continuous time processes through discrete sampling. Numerical results of simulation studies are presented and variance reduction problems are considered.

Introduction

Although the Black & Scholes framework is very simple, it is still a challenging task to efficiently price Asian options. Since we do not know explicitly the distribution of the arithmetic sum of log-normal variables, there is no closed form solution for the price of an Asian option. By the early nineties, many researchers attempted to address this problem and hence different approaches were studied including analytic approximations (see Turnball and Wakeman [20], Vorst [22], Levy [15] and more recently Lord [16]), PDE methods (see Vecer [21], Rogers and Shi [18], Ingersoll [11], Dubois and Lelievre [5]), Laplace transform inversion methods (see Geman and Yor [10], Geman and Eydeland [8]) and, of course, Monte Carlo simulation methods (see Kemna and Vorst [13], Broadie and Glasserman [3], Fu *et al.* [7]).

Monte Carlo simulation can be computationally expensive because of the usual statistical error. Variance reduction techniques are then essential to accelerate the convergence (one of the most efficient techniques is the Kemna&Vorst control variate based on the geometric average). One must also account for the inherent discretization bias resulting from approximating the continuous average of the stock price with a discrete one. It is crucial to choose with care the discretization scheme in order to have an accurate solution (see Lapeyre and Temam [14]). The main contribution of our work is to fully address this last feature by the use, after a suitable change of variables, of an exact simulation method inspired from the recent work of Beskos *et al.* [1, 2] and Fearnhead *et al.* [6].

In the first part of the paper, we recall the algorithm introduced by Beskos *et al.* [1] in order to simulate sample-paths of processes solving one-dimensional stochastic differential equations. By a suitable change of variables, one may suppose that the diffusion coefficient is equal to one. Then, according to the Girsanov theorem, one may deal with the drift coefficient by introducing an exponential martingale weight. Because of the one-dimensional setting, the stochastic integral in this exponential weight is equal to a standard

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integral with respect to the time variable up to the addition of a function of the terminal value of the path. Under suitable assumptions, conditionally on a Brownian path, an event with probability equal to the normalized exponential weight can be simulated using a Poisson point process. This allows to accept or reject this Brownian path as a path solution to the SDE with diffusion coefficient equal to one. In finance, one is interested in computing expectations rather than exact simulation of the paths. In this perspective, computation of the exponential importance sampling weight is enough. The entire series expansion of the exponential function permits to replace this exponential weight by a computable weight with the same conditional expectation given the Brownian path. This idea was first introduced by Wagner [23, 24, 25, 26] in a statistical physics context and it was very recently revisited by Beskos *et al.* [2] and Fearnhead *et al.* [6] for the estimation of partially observed diffusions. Some of the assumptions necessary to implement the exact algorithm of Beskos *et al.* [1] can then be weakened.

The second part is devoted to the application of these methods to option pricing within the Black & Scholes framework. Throughout the paper, $S_t = S_0 \exp\left(\sigma W_t + (r - \delta - \frac{\sigma^2}{2})t\right)$ represents the stock price at time t, T the maturity of the option, r the short interest rate, σ the volatility parameter, δ the dividend rate and $(W)_{t\in[0,T]}$ denotes a standard Brownian motion on the risk-neutral probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We are interested in computing the price $C_0 = \mathbb{E}\left(e^{-rT} f\left(\alpha S_T + \beta \int_0^T S_t dt\right)\right)$ of a European option with pay-off $f\left(\alpha S_T + \beta \int_0^T S_t dt\right)$ assumed to be square integrable under the risk neutral measure $\mathbb{P}$. The constants α and β are two given non-negative parameters.

When $\alpha > 0$, we remark that, by a change of variables inspired by Rogers and Shi [18], $\alpha S_T + \beta \int_0^T S_t dt$ has the same law as the solution at time T of a well-chosen one-dimensional stochastic differential equation. Then it is easy to implement the exact methods previously presented. The case $\alpha = 0$ of standard Asian options is more intricate. The previous approach does not work and we propose a new change of variables which is singular at initial time. It is not possible to implement neither the exact simulation algorithm nor the method based on the unbiased estimator of Wagner [23] and we propose a pseudo-exact hybrid method which appears as an extension of the exact simulation algorithm. In both cases, one first replaces the integral with respect to the time variable in the function f by an integral with respect to time in the exponential function. Because of the nice properties of this last function, exact computation is possible.

1 Exact Simulation techniques

1.1 The exact simulation method of Beskos *et al.* [1]

In a recent paper, Beskos *et al.* [1] proposed an algorithm which allows to simulate exactly the solution of a 1-dimensional stochastic differential equation. Under some hypotheses, they manage to implement an acceptance-rejection algorithm over the whole path of the solution, based on recursive simulation of a biased Brownian motion. Let us briefly recall their methodology. We refer to [1] for the demonstrations and a detailed presentation.

Consider the stochastic process $(\xi_t)_{0\leq t\leq T}$ determined as the solution of a general stochastic differential equation of the form :

$$\begin{cases} d\xi_t &= b(\xi_t)dt + \sigma(\xi_t)dW_t \\ \xi_0 &= \xi \in \mathbb{R} \end{cases} \tag{1}$$

where b and σ are scalar functions satisfying the usual Lipschitz and growth conditions with σ non vanishing. To simplify this equation, Beskos *et al.* [1] suggest to use the following change of variables : $X_t = \eta(\xi_t)$ where η is a primitive of $\frac{1}{\sigma}$ $(\eta(x) = \int_.^x \frac{1}{\sigma(u)} du)$.

Under the additional assumption that $\frac{1}{\sigma}$ is continuously differentiable, one can apply Itô's lemma to get

$$\begin{aligned} dX_t &= \eta'(\xi_t)d\xi_t + \frac{1}{2}\eta''(\xi_t)\, d<\xi,\xi>_t \\ &= \frac{b(\xi_t)}{\sigma(\xi_t)}dt + dW_t - \frac{\sigma'(\xi_t)}{2}dt \\ &= \underbrace{\left(\frac{b(\eta^{-1}(X_t))}{\sigma(\eta^{-1}(X_t))} - \frac{\sigma'(\eta^{-1}(X_t))}{2}\right)}_{a(X_t)} dt + dW_t \end{aligned}$$

So $\xi_t = \eta^{-1}(X_t)$ where $(X_t)_t$ is a solution of the stochastic differential equation

$$\left\{ \begin{array}{rcl} dX_t &=& a(X_t)dt + dW_t \\ X_0 &=& x. \end{array} \right. \tag{2}$$

Thus, without loss of generality, one can start from equation (2) instead of (1).

Let us denote by $(W_t^x)_{t\in[0,T]}$ the process $(W_t+x)_{t\in[0,T]}$, by $\mathbb{Q}_{W^x}$ its law and by $\mathbb{Q}_X$ the law of the process $(X_t)_{t\in[0,T]}$. From now on, we will denote by $(Y_t)_{t\in[0,T]}$ the canonical process, that is the coordinate mapping on the set $C([0,T],\mathbb{R})$ of real continuous maps on $[0,T]$ (see Revuz and Yor [17] or Karatzas and Shreve [12]).

One needs the following assumption to be true

Assumption 1 : *Under $\mathbb{Q}_{W^x}$, the process*

$$L_t = \exp\left[\int_0^t a(Y_u)dY_u - \frac{1}{2}\int_0^t a^2(Y_u)du\right]$$

is a martingale.

According to Rydberg [19] (see the proof of Proposition 4 where we give his argument on a specific example), a sufficient condition for this assumption to hold is

-Existence and uniqueness in law of a solution to the SDE (2).

-$\forall t \in [0,T]$, $\displaystyle\int_0^t a^2(Y_u)du < \infty$, $\mathbb{Q}_X$ and $\mathbb{Q}_{W^x}$ almost surely on $C([0,T],\mathbb{R})$.

Thanks to this assumption, one can apply the Girsanov theorem to get that $\mathbb{Q}_X$ is absolutely continuous with respect to $\mathbb{Q}_{W^x}$ and its Radon-Nikodym derivative is equal to

$$\frac{d\mathbb{Q}_X}{d\mathbb{Q}_{W^x}} = \exp\left[\int_0^T a(Y_t)dY_t - \frac{1}{2}\int_0^T a^2(Y_t)dt\right].$$

Consider A the primitive of the drift a, and assume that

Assumption 2 : *a is continuously differentiable.*

Since, by Itô's lemma, $A(W_T^x) = A(x) + \int_0^T a(W_t^x)dW_t^x + \frac{1}{2}\int_0^T a'(W_t^x)dt$, we have

$$\frac{d\mathbb{Q}_X}{d\mathbb{Q}_{W^x}} = \exp\left[A(Y_T) - A(x) - \frac{1}{2}\int_0^T a^2(Y_t) + a'(Y_t)dt\right].$$

Before setting up an acceptance-rejection algorithm using this Radon-Nikodym derivative, a last step is needed. To ensure the existence of a density $h(u)$ proportional to $\exp(A(u) - \frac{(u-x)^2}{2T})$, it is necessary and sufficient that the following assumption holds

Assumption 3 : *The function* $u \mapsto \exp(A(u) - \frac{(u-x)^2}{2T})$ *is integrable.*

Finally, let us define a process Z_t distributed according to the following law $\mathbb{Q}_Z$

$$\mathbb{Q}_Z = \int_{\mathbb{R}} \mathcal{L}\Big((W_t^x)_{t\in[0,T]} | W_T^x = y\Big) h(y)dy$$

where the notation $\mathcal{L}(.|.)$ stands for the conditional law. One has

$$\frac{d\mathbb{Q}_X}{d\mathbb{Q}_Z} = \frac{d\mathbb{Q}_X}{d\mathbb{Q}_{W^x}} \frac{d\mathbb{Q}_{W^x}}{d\mathbb{Q}_Z} = C \exp\left[-\frac{1}{2}\int_0^T a^2(Y_t) + a'(Y_t)dt\right]$$

where C is a normalizing constant. At this level, Beskos *et al.* [1] need another assumption

Assumption 4 : *The function* $\phi : x \mapsto \frac{a^2(x)+a'(x)}{2}$ *is bounded from below.*

Therefore, one can find a lower bound k of this function and eventually the Radon-Nikodym derivative of the change of measure between X and Z takes the form

$$\frac{d\mathbb{Q}_X}{d\mathbb{Q}_Z} = Ce^{-kT} \exp\left[-\int_0^T \phi(Y_t) - k\, dt\right].$$

The idea behind the exact algorithm is the following : suppose that one is able to simulate a continuous path $Z_t(\omega)$ distributed according to $\mathbb{Q}_Z$ and let $M(\omega)$ be an upper bound of the mapping $t \mapsto \phi(Z_t(\omega)) - k$. Let N be an independent random variable which follows the Poisson distribution with parameter $TM(\omega)$ and let $(U_i, V_i)_{i=1\ldots N}$ be a sequence of independent random variables uniformly distributed on $[0,T]\times[0,M(\omega)]$. Then, the number of points (U_i, V_i) which fall below the graph $\{(t, \phi(Z_t(\omega)) - k); t\in[0,T]\}$ is equal to zero with probability $\exp\left[-\int_0^T \phi(Z_t(\omega)) - k\, dt\right]$. Actually, simulating the whole path $(Z_t)_{t\in[0,T]}$ is not necessary. It is sufficient to determine an upper bound for $\phi(Z_t) - k$ since, as pointed out by the authors, it is possible to simulate recursively a Brownian motion on a bounded time interval by first simulating its endpoint, then simulating its minimum or its maximum and finally simulating the other points[2]. For this reason, one needs the following assumption for the algorithm to be feasible :

Assumption 5 : *Either* $\limsup_{u\to+\infty} \phi(u) < +\infty$ *or* $\limsup_{u\to-\infty} \phi(u) < +\infty$.

Suppose for example that $\limsup_{u\to+\infty} \phi(u) < +\infty$. The exact algorithm of Bekos *et al.* [1] then takes the following form :

Algorithm 1

1. *Draw the ending point* Z_T *of the process* Z *with respect to the density* h.
2. *Simulate the minimum* m *of the process* Z *given* Z_T.
3. *Fix an upper bound* $M(m) = \sup\{\phi(u) - k; u \geq m\}$ *for the mapping* $t \mapsto \phi(Z_t) - k$.
4. *Draw* N *according to the Poisson distribution with parameter* $TM(m)$ *and draw* $(U_i, V_i)_{i=1\ldots N}$, *a sequence of independent variables uniformly distributed on* $[0,T]\times[0,M(m)]$.
5. *Fill in the path of* Z *at the remaining times* $(U_i)_{i=1\ldots N}$.

[2] In their paper, the authors explain how to do such a decomposition of the Brownian path.

6. *Evaluate the number of points* $(V_i)_{i=1\ldots N}$ *such that* $V_i \leq \phi(Z_{U_i}) - k$.
If it is equal to zero, then return the simulated path Z.
Else, return to step 1.

This algorithm gives exact skeletons of the process X, solution of the SDE (2). Once accepted, a path can be further recursively simulated at additional times without any other acceptance/rejection criteria. We also point out that the same technique can be generalized by replacing the Brownian motion in the law of the proposal Z by any process that one is able to simulate recursively by first simulating its ending point, its minimum/maximum and then the other points. Also, the extension of the algorithm to the inhomogeneous case, where the drift coefficient a in (2), and therefore the function ϕ, depend on the time variable t, is straightforward given that the assumptions presented above are appropriately modified.

1.2 The unbiased estimator (U.E)

In finance, the pricing of contingent claims often comes down to the problem of computing an expectation of the form

$$C_0 = \mathbb{E}\left(f(X_T)\right) \tag{3}$$

where X is a solution of the SDE (2) and f is a scalar function such that $f(X_T)$ is square integrable. In a simulation based approach, one is usually unable to exhibit an explicit solution of this SDE and will therefore resort to numerical discretization schemes, such as the Euler or Milstein schemes, which introduce a bias. Of course, the exact algorithm presented above avoids this bias. Here, we are going to present a technique which permits to compute exactly the expectation (3) while assumptions 4 and 5 on the function $\frac{a^2+a'}{2}$ which appears in the Radon-Nikodym derivative are relaxed.

Using the previous results and notations, we get, under the assumptions 1 and 2, that

$$C_0 = \mathbb{E}\left(f(W_T^x)\exp\left[A(W_T^x) - A(x) - \frac{1}{2}\int_0^T a^2(W_t^x) + a'(W_t^x)dt\right]\right). \tag{4}$$

In order to implement an importance sampling method, let us introduce a positive density ρ on the real line and a process $(Z_t)_{t\in[0,T]}$ distributed according to the following law $\mathbb{Q}_Z$

$$\mathbb{Q}_Z = \int_{\mathbb{R}} \mathcal{L}\Big((W_t^x)_{t\in[0,T]}|W_T^x = y\Big)\rho(y)dy.$$

By (4), one has

$$C_0 = \mathbb{E}\left(\psi(Z_T)\exp\left[-\int_0^T \phi(Z_t)dt\right]\right) \tag{5}$$

where $\psi : z \mapsto f(z)\frac{e^{A(z)-A(x)-\frac{(z-x)^2}{2T}}}{\sqrt{2\pi}\rho(z)}$ and $\phi : z \mapsto \frac{a^2(z)+a'(z)}{2}$. We do not impose ρ to be equal to the density h of the previous section. It is a free parameter chosen in such a way that it reduces the variance of the simulation.

In his first paper, Wagner [23] constructs an unbiased estimator of the expectation (5) when ψ is a constant, $(Z_t)_{t\in[0,T]}$ is an $\mathbb{R}^d-$valued Markov process with known transition function and ϕ is a measurable function such that $\mathbb{E}\left(e^{\int_0^T|\phi(Z_t)|dt}\right) < +\infty$. His main idea is to expand the exponential term in a power series, then, using the transition function of the underlying Markov process and symmetry arguments, he constructs a signed measure ν on the space $\mathcal{Y} = \bigcup_{n=0}^{+\infty}([0,T]\times\mathbb{R}^d)^{n+1}$ such that the expectation at hand is equal to $\nu(\mathcal{Y})$. Consequently, any probability measure μ on Y that is absolutely continuous with respect to ν gives rise to an unbiased estimator ζ defined on $(\mathcal{Y},\mu)$ via $\zeta(y) = \frac{d\nu}{d\mu}(y)$. In practice, a suitable way to construct such an estimator is to use a Markov chain with an absorbing state. Wagner also discusses variance reduction techniques, specially importance sampling and a shift procedure consisting on adding a constant c

to the integrand ϕ and then multiplying by the factor e^{-cT} in order to get the right expectation. Wagner [25] extends the class of unbiased estimators by perturbing the integrand ϕ by a suitably chosen function ϕ_0 and then using mixed integration formulas representation. Very recently, Beskos *et al.* [2] obtained a simplified unbiased estimator for (5), termed Poisson estimator, using Wagner's idea of expanding the exponential in a power series and his shift procedure. To be specific, the Poisson estimator writes

$$\psi(Z_T)e^{c_p T - cT}\prod_{i=1}^{N}\frac{c-\phi(Z_{V_i})}{c_P} \tag{6}$$

where N is a Poisson random variable with parameter c_P and $(V_i)_i$ is a sequence of independent random variables uniformly distributed on $[0,T]$. Fearnhead *et al.* [6] generalized this estimator allowing c and c_P to depend on Z and N to be distributed according to any positive probability distribution on $\mathbb{N}$. They termed the new estimator the generalized Poisson estimator. We introduce a new degree of freedom by allowing the sequence $(V_i)_i$ to be distributed according to any positive density on $[0,T]$. This gives rise to the following unbiased estimator for (5) :

Lemma 1 — *Let p_Z and q_Z denote respectively a positive probability measure on $\mathbb{N}$ and a positive probability density on $[0,T]$. Let N be distributed according to p_Z and $(V_i)_{i\in\mathbb{N}^*}$ be a sequence of independent random variables identically distributed according to the density q_Z, both independent from each other conditionally on the process $(Z_t)_{t\in[0,T]}$. Let c_Z be a real number which may depend on Z. Assume that*

$$\mathbb{E}\left(|\psi(Z_T)|e^{-c_Z T}\exp\left[\int_0^T |c_Z-\phi(Z_t)|dt\right]\right)<\infty.$$

Then

$$\psi(Z_T)e^{-c_z T}\frac{1}{p_Z(N)\,N!}\prod_{i=1}^{N}\frac{c_Z-\phi(Z_{V_i})}{q_Z(V_i)} \tag{7}$$

is an unbiased estimator of C_0.

Proof : The result follows from

$$\begin{aligned}\mathbb{E}\left(\psi(Z_T)e^{-c_z T}\frac{1}{p_Z(N)\,N!}\prod_{i=1}^{N}\frac{c_Z-\phi(Z_{V_i})}{q_Z(V_i)}\Big|(Z_t)_{t\in[0,T]}\right) &= \psi(Z_T)e^{-c_z T}\sum_{n=0}^{+\infty}\frac{\left(\int_0^T c_Z-\phi(Z_t)dt\right)^n}{p_Z(n)\,n!}\,p_Z(n)\\ &= \psi(Z_T)\exp\left(-\int_0^T \phi(Z_t)dt\right).\end{aligned}$$

□

Using (7), one is now able to compute the expectation at hand by a simple Monte Carlo simulation. The practical choice of p_Z and q_Z conditionally on Z is studied in the appendix 4.1.

As pointed out in Fearnhead *et al.* [6], this method is an extension of the exact algorithm method since, under assumptions 3, 4 and 5, the reinforced integrability assumption of Lemma 1 is always satisfied.

Indeed, suppose for example that $\limsup_{u\to+\infty}\phi(u)<+\infty$ and let k be a lower bound of ϕ, m_Z be the minimum of the process Z and M_Z an upper bound of $\{\phi(u)-k, u\geq m_Z\}$. Then, taking $c_Z=M_Z+k$ in

Lemma 1 ensures the integrability condition :

$$\begin{aligned}
\mathbb{E}\left(|\psi(Z_T)|e^{-(M_Z+k)T}e^{\int_0^T |M_Z+k-\phi(Z_t)|dt}\right) &= \mathbb{E}\left(|\psi(Z_T)|e^{-(M_Z+k)T}e^{\int_0^T M_Z+k-\phi(Z_t)dt}\right)\\
&= \mathbb{E}\left(|\psi(Z_T)|e^{-\int_0^T \phi(Z_t)dt}\right) < \infty
\end{aligned}$$

and hence, one is allowed to write that

$$C_0 = \mathbb{E}\left(\psi(Z_T)e^{-(M_Z+k)T}\frac{1}{p_Z(N)N!}\prod_{i=1}^{N}\frac{M_Z+k-\phi(Z_{V_i})}{q_Z(V_i)}\right).$$

Better still, the random variable $\psi(Z_T)e^{-(M_Z+k)T}\frac{1}{p_Z(N)N!}\prod_{i=1}^{N}\frac{M_Z+k-\phi(Z_{V_i})}{q_Z(V_i)}$ is square integrable when p_Z is the Poisson distribution with parameter $M_Z T + k$ and q_Z is the uniform distribution on $[0,T]$ since we have then

$$\begin{aligned}
\mathbb{E}\left(\left(\psi(Z_T)e^{-(M_Z+k)T}\frac{1}{p_Z(N)N!}\prod_{i=1}^{N}\frac{M_Z+k-\phi(Z_{V_i})}{q_Z(V_i)}\right)^2\right) &= \mathbb{E}\left(\psi^2(Z_T)\prod_{i=1}^{N}\left(1-\frac{\phi(Z_{V_i})}{M_Z+k}\right)^2\right)\\
&\leq \mathbb{E}\left(\psi^2(Z_T)\right) < \infty.
\end{aligned}$$

The last inequality follows from the square integrability of f : whenever one is able to simulate from the density h, introduced in the exact algorithm, by doing rejection sampling, there exists a density ρ such that ψ, which is equal to $f(Z_T)\frac{h(Z_T)}{\rho(Z_T)}$ up to a constant factor, is dominated by f and so is square integrable.

The square integrability property is very important in that we use a Monte Carlo method. We see that, whenever the exact algorithm is feasible, the unbiased estimator of lemma 1 is a simulable square integrable random variable, at least for the previous choice of p_Z and q_Z.

Remark 2 — *One can derive two estimators of C_0 from the result of Lemma 1 :*

$$\begin{aligned}
\delta_1 &= \frac{1}{n}\sum_{i=1}^{n} f(Z_T^i)\frac{e^{A(Z_T^i)-A(x)-\frac{(Z_T^i-x)^2}{2T}}}{\sqrt{2\pi}\rho(Z_T^i)}e^{-c_Z T}\frac{1}{p_Z(N^i)\,N^i!}\prod_{j=1}^{N^i}\frac{c_Z-\phi(Z_{V_j^i})}{q_Z(V_j^i)}\\
\delta_2 &= \frac{\displaystyle\sum_{i=1}^{n} f(Z_T^i)\frac{e^{A(Z_T^i)-A(x)-\frac{(Z_T^i-x)^2}{2T}}}{\sqrt{2\pi}\rho(Z_T^i)}\frac{1}{p_Z(N^i)\,N^i!}\prod_{j=1}^{N^i}\frac{c_Z-\phi(Z_{V_j^i})}{q_Z(V_j^i)}}{\displaystyle\sum_{i=1}^{n}\frac{e^{A(Z_T^i)-A(x)-\frac{(Z_T^i-x)^2}{2T}}}{\sqrt{2\pi}\rho(Z_T^i)}\frac{1}{p_Z(N^i)\,N^i!}\prod_{j=1}^{N^i}\frac{c_Z-\phi(Z_{V_j^i})}{q_Z(V_j^i)}}.
\end{aligned}$$

2 Application : the pricing of continuous Asian options

In the Black & Scholes model, the stock price is the solution of the following SDE under the risk-neutral measure $\mathbb{P}$

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sigma dW_t \tag{8}$$

where all the parameters are constant : r is the short interest rate, δ is the dividend rate and σ is the volatility.

Throughout, we denote $\gamma = r - \delta - \frac{\sigma^2}{2}$. The path-wise unique solution of (8) is

$$S_t = S_0\,\exp(\sigma W_t + \gamma t)\,.$$

We consider an option with pay-off of the form

$$f\left(\alpha S_T + \beta \int_0^T S_t dt\right) \tag{9}$$

where f is a given function such that $\mathbb{E}\left(f^2\left(\alpha S_T + \beta \int_0^T S_t dt\right)\right) < \infty$, T is the maturity of the option and α, β are two given non negative parameters[3]. Note that for $\alpha = 0$, this is the pay-off of a standard continuous Asian option.

The fundamental theorem of arbitrage-free pricing ensures that the price of the option under consideration is

$$C_0 = \mathbb{E}\left(e^{-rT} f\left(\alpha S_T + \beta \int_0^T S_u du\right)\right).$$

At first sight, the problem seems to involve two variables : the stock price and the integral of the stock price with respect to time. Dealing with the PDE associated with Asian option pricing, Rogers and Rogers and Shi [18] used a suitable change of variables to reduce the spatial dimension of the problem to one. We are going to use a similar idea.

Let

$$\xi_t = \left(\alpha S_0 + \beta S_0 \int_0^t e^{-\sigma W_u - \gamma u} du\right) e^{\sigma W_t + \gamma t}.$$

We have that

$$\begin{aligned} \xi_t &= \alpha S_0 e^{\sigma W_t + \gamma t} + \beta S_0 \int_0^t e^{\sigma(W_t - W_u) + \gamma(t-u)} du \\ &= \alpha S_0 e^{\sigma B_t + \gamma t} + \beta S_0 \int_0^t e^{\sigma B_s + \gamma s} ds \end{aligned}$$

where we set $B_s = W_t - W_{t-s}, \forall s \in [0,t]$. Clearly, $(B_s)_{s\in[0,t]}$ is a Brownian motion and thus the following lemma holds

Lemma 3 — $\forall t \in [0,T]$, ξ_t *and* $\alpha S_t + \beta \int_0^t S_u du$ *have the same law.*

As a consequence

$$C_0 = \mathbb{E}\left(e^{-rT} f(\xi_T)\right).$$

By applying Itô's lemma, we verify that the process $(\xi_t)_{t\geq 0}$ is a positive solution of the following 1-dimensional stochastic differential equation for which path-wise uniqueness holds

$$\left\{ \begin{array}{rcl} d\xi_t &=& \beta S_0 dt + \xi_t(\sigma dW_t + (\gamma + \frac{\sigma^2}{2})dt) \\ \xi_0 &=& \alpha S_0. \end{array} \right. \tag{10}$$

We are thus able to value C_0 by Monte Carlo simulation without resorting to discretization schemes using one of the exact simulation techniques described in the previous section. In the case $\alpha = 0$, one has to deal with the fact that ξ_t starts from zero which is the reason why we distinguish two cases.

[3]The underlying of this option is a weighted average of the stock price at maturity and the running average of the stock price until maturity with respective weights α and βT.

2.1 The case $\alpha \neq 0$

We are going to apply both the exact algorithm of Beskos *et al.* [1] and the method based on the unbiased estimator of lemma 1.

We make the following change of variables to have a diffusion coefficient equal to 1 :

$$X_t = \frac{\log(\xi_t)}{\sigma} \Rightarrow \left\{ \begin{array}{rcl} dX_t & = & (\frac{\gamma}{\sigma} + \frac{\beta S_0}{\sigma} e^{-\sigma X_t})dt + dW_t \\ X_0 & = & x \quad \text{with } x = \frac{\log(\alpha S_0)}{\sigma}. \end{array} \right. \tag{11}$$

Thus

$$C_0 = \mathbb{E}\left(e^{-rT} f(e^{\sigma X_T})\right).$$

The following proposition ensures that assumption 1 is satisfied.

Proposition 4 — *The process $(L_t)_{t\in[0,T]}$ defined by*

$$L_t = \exp\left[\int_0^T (\frac{\gamma}{\sigma} + \frac{\beta S_0}{\sigma} e^{-\sigma Y_t})\, dY_t - \frac{1}{2}\int_0^T (\frac{\gamma}{\sigma} + \frac{\beta S_0}{\sigma} e^{-\sigma Y_t})^2 dt\right]$$

is a martingale under $\mathbb{Q}_{W^x}$.

Proof : Under $\mathbb{Q}_{W^x}$, $(L_t)_{t\in[0,T]}$ is clearly a non-negative local martingale and hence a super-martingale. Then, it is a true martingale if and only if $\mathbb{E}_{\mathbb{Q}_{W^x}}(L_T) = 1$.

Checking the classical Novikov's or Kamazaki's criteria is not straightforward. Instead, we are going to use the approach developed by Rydberg [19] (see also Wong and Heyde [27]) who takes advantage of the link between explosions of SDEs and the martingale property of stochastic exponentials.

Let us define the following stopping times :

$$\tau_n(Y) = \inf\left\{ t \in \mathbb{R}^+ \text{ such that } \int_0^t \left(\frac{\gamma}{\sigma} + \frac{\beta S_0}{\sigma} e^{-\sigma Y_u}\right)^2 du \geq n \right\},$$

with the convention $\inf\{\emptyset\} = +\infty$.

The stopped process $(L_{t\wedge\tau_n(Y)})_{t\in[0,T]}$ is a true martingale under $\mathbb{Q}_{W^x}$ since Novikov's condition is fulfilled. According to the Girsanov theorem, one can define a new probability measure $\mathbb{Q}_X^n$, which is absolutely continuous with respect to $\mathbb{Q}_{W^x}$, by its Radon-Nikodym derivative

$$\frac{d\mathbb{Q}_X^n}{d\mathbb{Q}_{W^x}} = L_{T\wedge\tau_n(Y)}.$$

Hence

$$\mathbb{E}_{\mathbb{Q}_X^n}\left(\mathbb{1}_{\{\tau_n(Y)>T\}}\right) = \mathbb{E}_{\mathbb{Q}_{W^x}}\left(\mathbb{1}_{\{\tau_n(Y)>T\}} L_{T\wedge\tau_n(Y)}\right).$$

Since $(\tau_n(Y))_{n\in\mathbb{N}}$ is a non decreasing sequence, we can pass to the limit in the right hand side We get

$$\lim_{n\to+\infty} \mathbb{Q}_X^n\left(\tau_n(Y) > T\right) = \mathbb{E}_{\mathbb{Q}_{W^x}}\left(\mathbb{1}_{\{\tau_\infty(Y)>T\}} L_{T\wedge\tau_\infty(Y)}\right)$$

where $\tau_\infty(Y)$ denotes the limit of the non decreasing sequence $(\tau_n(Y))_{n\in\mathbb{N}}$.

Under $\mathbb{Q}_{W^x}$, $(Y_t)_{t\in[0,T]}$ has the same law as a Brownian motion starting from x so $\tau_\infty(Y) = +\infty$, $\mathbb{Q}_{W^x}$ almost surely, and consequently

$$\mathbb{E}_{\mathbb{Q}_{W^x}}\left(L_T\right) = \lim_{n\to+\infty} \mathbb{Q}_X^n\left(\tau_n(Y) > T\right).$$

On the other hand, the Girsanov theorem implies that, under $\mathbb{Q}_X^n$, $(Y_t)_{t\in[0,T\wedge\tau_n(Y)]}$ solves a SDE of the form (11). To conclude the proof, it is sufficient to check that trajectorial uniqueness holds for this SDE. Indeed, the law of $(Y_t)_{t\in[0,T\wedge\tau_n(Y)]}$ under $\mathbb{Q}_X^n$ is the same as the law of $(Y_t)_{t\in[0,T\wedge\tau_n(Y)]}$ under $\mathbb{Q}_X$. Hence

$$\mathbb{Q}_X^n\left(\tau_n(Y) > T\right) = \mathbb{Q}_X\left(\tau_n(Y) > T\right) \underset{n\to+\infty}{\longrightarrow} \mathbb{Q}_X\left(\tau_\infty(Y) > T\right).$$

Clearly, $\int_0^t \left(\frac{\gamma}{\sigma} + \frac{\beta S_0}{\sigma} e^{-\sigma Y_u}\right)^2 du < \infty$, $\mathbb{Q}_X$ almost surely, so

$$\mathbb{E}_{\mathbb{Q}_{W^x}}\left(L_T\right) = \mathbb{Q}_X\left(\tau_\infty(Y) > T\right) = 1$$

as required.

In order to check trajectorial uniqueness for the SDE (11), we consider two solutions X^1 and X^2. We have that

$$d(X_t^1 - X_t^2) = \frac{\beta S_0}{\sigma}\left(e^{-\sigma X_t^1} - e^{-\sigma X_t^2}\right) dt \Rightarrow d|X_t^1 - X_t^2| = \frac{\beta S_0}{\sigma} sign(X_t^1 - X_t^2)\left(e^{-\sigma X_t^1} - e^{-\sigma X_t^2}\right) dt.$$

So

$$|X_t^1 - X_t^2| = \frac{\beta S_0}{\sigma}\int_0^t sign(X_s^1 - X_s^2)\left(e^{-\sigma X_t^1} - e^{-\sigma X_t^2}\right) ds \leq 0.$$

The last inequality follows from the fact that $x \mapsto e^{-\sigma x}$ is a decreasing function. Finally, almost surely, $\forall t \geq 0$, $X_t^1 = X_t^2$ which leads to strong uniqueness. □

Consequently, thanks to the Girsanov theorem, we have

$$\frac{d\mathbb{Q}_X}{d\mathbb{Q}_{W^x}} = \exp\left[\int_0^T \underbrace{\left(\frac{\gamma}{\sigma} + \frac{\beta S_0}{\sigma} e^{-\sigma Y_t}\right)}_{a(Y_t)} dY_t - \frac{1}{2}\int_0^T \left(\frac{\gamma}{\sigma} + \frac{\beta S_0}{\sigma} e^{-\sigma Y_t}\right)^2 dt\right]. \tag{12}$$

Set $A(u) = \int_0^u a(x)dx = \frac{\gamma}{\sigma}u + \frac{\beta S_0}{\sigma^2}(1 - e^{-\sigma u})$. Then

$$\frac{d\mathbb{Q}_X}{d\mathbb{Q}_{W^x}} = \exp\left[A(Y_T) - A(x) - \frac{1}{2}\int_0^T a^2(Y_t) + a'(Y_t)dt\right].$$

The function $u \mapsto \exp\left(A(u) - \frac{(u-Y_0)^2}{2T}\right) = \exp\left(\frac{\gamma}{\sigma}u + \frac{\beta S_0}{\sigma^2}(1 - e^{-\sigma u}) - \frac{(u-Y_0)^2}{2T}\right)$ is clearly integrable so we can define a new process $(Z_t)_{t\in[0,T]}$ distributed according to the following law $\mathbb{Q}_Z$

$$\mathbb{Q}_Z = \int_{\mathbb{R}} \mathcal{L}\Big((W_t)_{t\in[0,T]} | W_T = y\Big) h(y)dy$$

where the probability density h is of the form

$$h(u) = C\exp\left(A(u) - \frac{(u-Y_0)^2}{2T}\right) \quad \text{with } C \text{ a normalizing constant.} \tag{13}$$

Remark 5 — *Simulating from this probability distribution is not difficult (see the appendix 4.2 for an appropriate method of acceptance/rejection sampling).*

We have

$$\frac{d\mathbb{Q}_X}{d\mathbb{Q}_Z} = C \exp\left[-\int_0^T \frac{1}{2}(a^2(Y_t) + a'(Y_t))dt\right].$$

Set $\phi(x) = \frac{a^2(x)+a'(x)}{2} = \frac{(\frac{\gamma}{\sigma}+\frac{\beta S_0}{\sigma}e^{-\sigma x})^2 - \beta S_0 e^{-\sigma x}}{2}$. A direct calculation gives

$$\inf_{x\in\mathbb{R}} \phi(x) = \begin{cases} \frac{\gamma^2}{2\sigma^2} & \text{if } 2\gamma \geq \sigma^2 \\ \phi\left(\frac{1}{\sigma}\log(\frac{2\beta S_0}{\sigma^2-2\gamma})\right) & \text{otherwise.} \end{cases}$$

Set $k = \inf_{x\in\mathbb{R}} \phi(x)$. Finally, we get

$$\frac{d\mathbb{Q}_X}{d\mathbb{Q}_Z} = Ce^{-kT} \exp\left[-\int_0^T \phi(Y_t) - k\, dt\right].$$

We check that

$$\lim_{x\to+\infty} \phi(x) = \frac{\gamma^2}{2\sigma^2} < \infty$$
$$\lim_{x\to-\infty} \phi(x) = +\infty.$$

Hence we can apply the algorithm 1 to simulate exactly X_T and compute $C_0 = \mathbb{E}\left(e^{-rT}f(e^{\sigma X_T})\right)$ by Monte Carlo. On the other hand, using (12) we get

$$C_0 = \mathbb{E}\left(e^{-rT}f(e^{\sigma W_T^x})\exp\left[A(W_T^x) - A(x) - \frac{1}{2}\int_0^T a^2(W_t^x) + a'(W_t^x)dt\right]\right)$$

and we can also use the unbiased estimator presented in the previous section to compute this expectation.

Remark 6 — *We also applied the exact algorithm based on a geometric Brownian motion instead of the standard Brownian motion which seems more intuitive given the form of the SDE (10). The algorithm is feasible because we can simulate recursively a drifted Brownian motion and therefore a geometric Brownian motion by an exponential change of variables. The results we obtained were not different from the first method.*

2.1.1 Numerical computation

For numerical tests, we consider the case

$$f(x) = (x-K)_+$$

which corresponds to the European call option with strike K. Using the exact simulation algorithm presented above, we can simulate the underlying $\alpha S_T + \beta \int_0^T S_t dt$ at maturity (see Figure 1). Then, all we have to do is a simple Monte Carlo method to get the price of the option under consideration. Using the unbiased estimator, we get

$$C_0 = \mathbb{E}\left(e^{-rT}(e^{\sigma Z_T} - K)_+ \frac{e^{A(Z_T)-A(x)-\frac{(Z_T-x)^2}{2T}}}{\sqrt{2\pi}\rho(Z_T)} e^{-(M_Z+k)T}\frac{1}{p_Z(N)N!}\prod_{i=1}^N \frac{M_Z + k - \phi(Z_{V_i})}{q(V_i)}\right)$$

where $(Z_t)_{t\in[0,T]}, \rho, M_Z, k, p_Z$ and q_Z are defined as in section 1.2. In order to ensure square integrability, we choose p_Z to be a Poisson distribution with parameter $M_Z T + k$ and q_Z to be the uniform distribution on

$[0, T]$. For the density ρ, a good choice is to consider the density that we use to simulate from the distribution h by rejection sampling.

We test these exact methods against a standard discretization scheme with the variance reduction technique of Kemna and Vorst [13]. As pointed out by Lapeyre and Temam [14], the discretization of the integral by a simple Riemannian sum is not efficient. Instead, we use the trapezoidal discretization. In the sequel, we will denote this method by Trap+KV. The table 1 gives the results we obtained for the following arbitrary set of parameters : $S_0 = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.3$, $\delta = 0$, $T = 1$, $\alpha = 0.6$ and $\beta = 0.4$. The computation has been made on a computer with a 2.8 Ghz Intel Penthium 4 processor. We intentionally choose a large number of simulations in order to show the influence of the number of time steps when using a discretization scheme.

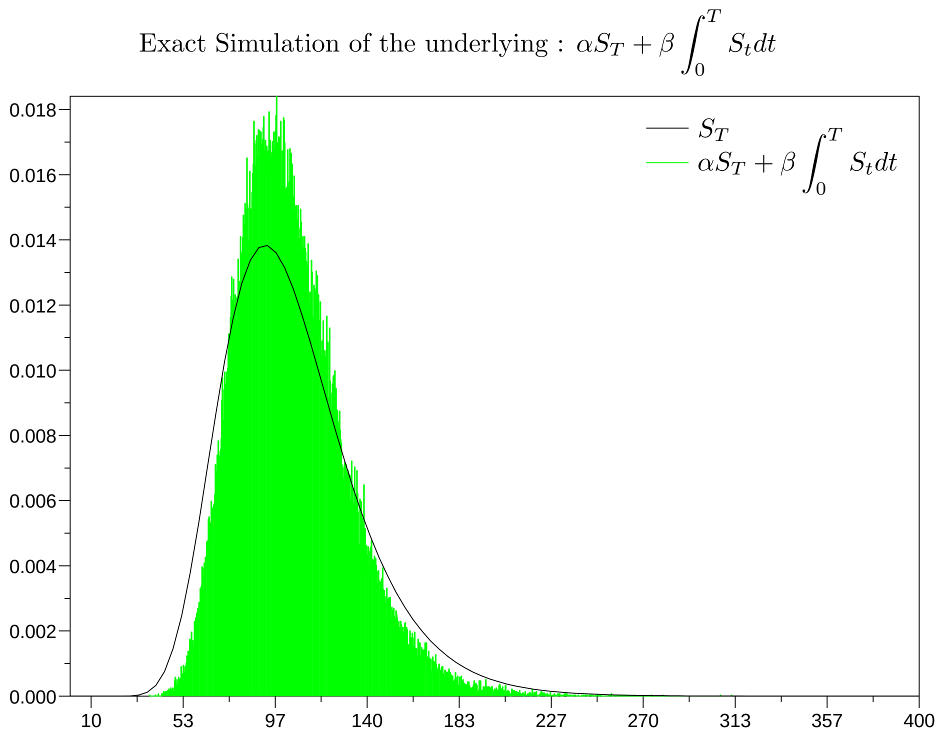

Figure 1: Histogram of 10^5 independent realizations of $\alpha S_T + \beta \int_0^T S_t dt$ for $\alpha = 0.6$ and $\beta = 0.4$ compared with the lognormal distribution of S_T.

Method	M	N	Acceptance rate	Price	C.I at 95%	CPU
Trap+KV	10	10^6	-	11.46	$[11.43, 11.48]$	5 s
	20			11.46	$[11.43, 11.49]$	9 s
	50			11.47	$[11.44, 11.5]$	21 s
Exact Simulation	-	10^6	24%	11.46	$[11.43, 11.5]$	81 s
U.E ($c_P = M_Z, c_Z = M_Z + k$)	-	10^6	-	11.46	$[11.43, 11.49]$	17 s
U.E ($c_P = c_Z = 1/T$)	-	10^6	-	11.46	$[11.43, 11.49]$	6 s

Table 1: Price of the option (9) using a standard discretization technique and exact simulation methods.

Empirical evidence shows that the exact simulation method is quite slow. This is mainly due to the fact that the rejection algorithm has a little acceptance rate (24% according to table 1). Using a geometric Brownian motion instead of a standard Brownian motion did not improve the results. Also, simulating recursively a Brownian path conditionally on its terminal value and its minimum is time consuming.

The unbiased estimator is more efficient, especially when we can avoid the recursive simulation of the Brownian path. To do so, we choose for p_Z a Poisson distribution with mean $c_P T$ where c_P is a free parameter. If we assume that the integrability condition in lemma 1 holds, then we can write that

$$C_0 = \mathbb{E}\left(e^{-rT}(e^{\sigma Z_T} - K)_+ \frac{e^{A(Z_T)-A(x)-\frac{(Z_T-x)^2}{2T}}}{\sqrt{2\pi}\rho(Z_T)} e^{c_P T - c_Z T} \prod_{i=1}^{N} \frac{c_Z - \phi(Z_{V_i})}{c_P} \right).$$

Regarding the dependence of the exact simulation method with respect to the parameters α and β, it is intuitive that whenever $\alpha >> \beta$, the method performs well since the logarithm of the underlying is not far from the logarithm of the geometric Brownian motion on which we do rejection-sampling. The table 2 confirms this intuition. We see that we cannot apply the algorithm for small values of α and then let $\alpha \to 0$ to treat the case $\alpha = 0$.

$\boldsymbol{\frac{\alpha}{\alpha+\beta}}$	0.3	0.4	0.5	0.6	0.7
Acceptance Rate	0.003%	0.47%	5.66%	24.43%	53.85%

Table 2: Influence of the parameter $\frac{\alpha}{\alpha+\beta}$ on the acceptance rate of the exact algorithm.

2.2 Standard Asian options : the case $\alpha = 0$ and $\beta > 0$

A standard Asian option is a European option on the average of the stock price over a determined period until maturity. An Asian call, for example, has a pay-off of the form $(\frac{1}{T}\int_0^T S_u du - K)_+$. With our previous notations, it corresponds to the case $\alpha = 0$, $\beta = \frac{1}{T}$ and $f(x) = (x-K)_+$.

The change of variables we used above is no longer suitable because it starts from zero when $\alpha = 0$. Instead, we consider the following new definition of the process ξ

$$\begin{cases} \xi_t &= \dfrac{S_0}{t}\displaystyle\int_0^t e^{\sigma(W_t - W_u)+\gamma(t-u)} du \\ \xi_0 &= S_0. \end{cases} \tag{14}$$

Obviously, the two variables ξ_T and $\frac{1}{T}\int_0^T S_u du$ have the same law. Hence, the price of the Asian option becomes

$$C_0 = \mathbb{E}\left(e^{-rT} f\left(\frac{1}{T}\int_0^T S_u du \right)\right) = \mathbb{E}\left(e^{-rT} f(\xi_T)\right).$$

Remark 7 — *The pricing of floating strike Asian options is also straightforward using this method. It is even more natural to consider these options since it unveils the appropriate change of variables as we shall see below.*

Let us consider a floating strike Asian call for example. We have to compute

$$C_0 = \mathbb{E}\left(e^{-rT}\left(\frac{1}{T}\int_0^T S_u du - S_T\right)_+ \right).$$

Using $\widetilde{S}_t = S_t e^{\delta t}$ as a numéraire (see the seminal paper of Geman et al. [9]), we immediately obtain that

$$C_0 = \mathbb{E}_{\mathbb{P}_{\widetilde{S}}} \left(S_0 e^{-\delta T} \big(\frac{1}{T} \int_0^T \frac{S_u}{S_T} du - 1 \big)_+ \right)$$

where $\mathbb{P}_{\widetilde{S}}$ is the probability measure associated to the numéraire $\widetilde{S}_t$. It is defined by its Radon-Nikodym derivative $\frac{d\mathbb{P}_{\widetilde{S}}}{d\mathbb{P}} = e^{\sigma W_T - \frac{\sigma^2}{2}T}$.

Under $\mathbb{P}_{\widetilde{S}}$, the process $B_t = W_t - \sigma t$ is a Brownian motion and we can write that

$$\begin{aligned} C_0 &= \mathbb{E}_{\mathbb{P}_{\widetilde{S}}} \left(S_0 e^{-\delta T} \big(\tfrac{1}{T} \int_0^T e^{\sigma(B_u - B_T) + (r - \delta + \frac{\sigma^2}{2})(u-T)} du - 1 \big)_+ \right) \\ &= \mathbb{E} \left(S_0 e^{-\delta T} \big(\tfrac{1}{T} \int_0^T e^{\sigma(W_u - W_T) + (r - \delta + \frac{\sigma^2}{2})(u-T)} du - 1 \big)_+ \right) \\ &= \mathbb{E} \left(e^{-\delta T} \big(\xi_T - S_0 \big)_+ \right) \end{aligned}$$

where ξ_t is the process defined by (14) but with $\gamma = r - \delta + \frac{\sigma^2}{2}$. We see therefore that the problem simplifies to the fixed strike Asian pricing problem.

Let us write down the stochastic differential equation that rules the process $(\xi_t)_{t \in [0,T]}$. Using Itô's lemma, we get

$$\left\{ \begin{array}{rcl} d\xi_t &=& \frac{\xi_0 - \xi_t}{t} dt + \xi_t \left(\sigma dW_t + (\gamma + \frac{\sigma^2}{2}) dt \right) \\ \xi_0 &=& S_0. \end{array} \right.$$

Note that we are faced with a singularity problem near 0 because of the term $\frac{\xi_0 - \xi_t}{t}$. We are going to reduce its effect using another change of variables.

Using Itô's lemma, we show that

$$C_0 = \mathbb{E} \left(e^{-rT} f \left(S_0 e^{X_T} \right) \right) \tag{15}$$

where $X_t = \log(\xi_t / \xi_0)$ solves the following SDE

$$\left\{ \begin{array}{rcl} dX_t &=& \sigma dW_t + \gamma dt + \frac{e^{-X_t} - 1}{t} dt \\ X_0 &=& 0. \end{array} \right. \tag{16}$$

Lemma 8 — *Existence and strong uniqueness hold for the stochastic differential equation (16).*

Proof : Existence is obvious since we have a particular solution X_t. The diffusion coefficient being constant and the drift coefficient being a decreasing function in the spatial variable, we have also strong uniqueness for the SDE (see the proof of Proposition 4). □

Because of the singularity of the term $\frac{e^{-X_t} - 1}{t}$ in the drift coefficient, the law of $(X_t)_{t \geq 0}$ is not absolutely continuous with respect to the law of $(\sigma W_t)_{t \geq 0}$. That is why we now define $(Z_t)_{t \geq 0}$ by the following SDE with an affine inhomogeneous drift coefficient :

$$\left\{ \begin{array}{rcl} dZ_t &=& \sigma dW_t + \gamma dt - \dfrac{Z_t}{t} dt \\ Z_0 &=& X_0 = 0. \end{array} \right. \tag{17}$$

The drift coefficient exhibits the same behavior as the one in (16) in the limit $t \to 0$ in order to ensure the desired absolute continuity property. It is affine in the spatial variable so that $(Z_t)_{t \geq 0}$ is a Gaussian process and as such is easy to simulate recursively.

Lemma 9 — *The process*

$$Z_t = \frac{\sigma}{t}\int_0^t s dW_s + \frac{\gamma}{2}t \tag{18}$$

is the unique solution of the stochastic differential equation (17).

Proof : Using Itô's Lemma, we easily check that Z_t given by (18) is a solution of (17). Again, constant diffusion coefficient and decreasing drift coefficient ensures strong uniqueness. □

Remark 10 — *For the computation of the price $C_0 = \mathbb{E}\left(e^{-rT}(S_0 e^{X_T} - K)_+\right)$ of a standard Asian call option, the random variable $e^{-rT}(S_0 e^{Z_T} - K)_+$ provides a natural control variate. Indeed, since Z_T is a Gaussian random variable with mean $\frac{\gamma}{2}T$ and variance $\frac{\sigma^2 T}{3}$, one has*

$$\mathbb{E}\left(e^{-rT}(S_0 e^{Z_T} - K)_+\right) = S_0 e^{(\frac{\gamma}{2}+\frac{\sigma^2}{6}-r)T}\mathcal{N}\left(d + \sigma\sqrt{\frac{1}{3}T}\right) - Ke^{-rT}\mathcal{N}(d)$$

where $\mathcal{N}$ is the cumulative standard normal distribution function and $d = \frac{\log(S_0/K)+\frac{\gamma}{2}T}{\sigma\sqrt{\frac{1}{3}T}}$.

Notice that in Kemna and Vorst [13], the authors suggest the use of the control variate $e^{-rT}\left(S_0 \exp\left(\frac{1}{T}\int_0^T \sigma W_t + \gamma t\, dt\right) - K\right)_+$ which has the same law than $e^{-rT}\left(S_0 e^{Z_T} - K\right)_+$ as $\frac{1}{T}\int_0^T \sigma W_t + \gamma t\, dt$ is also a Gaussian variable with mean $\frac{\gamma}{2}T$ and variance $\frac{\sigma^2 T}{3}$.

In order to define a new probability measure under which $(Z_t)_{t\geq 0}$ solves the SDE (16), one introduces

$$L_t = \exp\left[\int_0^t \frac{e^{-Z_s} - 1 + Z_s}{\sigma s} dW_s - \frac{1}{2}\int_0^t \left(\frac{e^{-Z_s} - 1 + Z_s}{\sigma s}\right)^2 ds\right].$$

Because of the singularity of the coefficients in the neighborhood of $s = 0$, one has to check that the integrals in L_t are well defined. This relies on the following lemma

Lemma 11 — *Let $\epsilon > 0$. In a random neighborhood of $s = 0$, we have*

$$|Z_s| \leq cs^{\frac{1}{2}-\epsilon} \text{ and } |X_s| \leq cs^{\frac{1}{2}-\epsilon}$$

where c is a constant depending on σ,γ and ϵ.

Since $\forall \epsilon > 0$,

$$\forall z \leq cs^{\frac{1}{2}-\epsilon}, \left(\frac{e^{-z} - 1 + z}{\sigma s}\right)^2 \leq Cs^{-4\epsilon},$$

we can choose $\epsilon < \frac{1}{4}$ to deduce that L_t is well defined.

Proof : We easily check that the Gaussian process $(B_t)_{t\in[0,T]}$ defined by $B_t = \int_0^{(3t)^{\frac{1}{3}}} sdW_s$ is a standard Brownian motion. Thanks to the law of iterated logarithm for the Brownian motion (see for example Karatzas and Shreve [12] p. 112), there exists $t_1(\omega)$ such that[4],

$$\forall t \leq t_1(\omega), |B_t(\omega)| \leq t^{\frac{1}{2}-\frac{\epsilon}{3}}.$$

[4] ω is an element of the underlying probability space Ω.

Therefore,

$$\forall t \leq (3t_1(\omega))^{\frac{1}{3}}, \quad |Z_t(\omega)| = \left|\frac{\sigma}{t} B_{\frac{t^3}{3}}(\omega) + \frac{\gamma}{2} t\right| \leq \frac{\sigma}{3^{\frac{1}{2}-\frac{\epsilon}{3}}} t^{\frac{1}{2}-\epsilon} + \frac{\gamma}{2} t.$$

Taking $c = \max(\frac{\sigma}{3^{\frac{1}{2}-\frac{\epsilon}{3}}}, \frac{\gamma}{2})$ yields

$$\forall t \leq (3t_1(\omega))^{\frac{1}{3}} \wedge 1, \quad |Z_t(\omega)| \leq c t^{\frac{1}{2}-\epsilon}.$$

On the other hand, recall that $X_t = \log(\xi_t/\xi_0) = \log\left(\frac{1}{t} e^{\sigma W_t + \gamma t} \int_0^t e^{-\sigma W_u - \gamma u} du\right)$. So, using the law of iterated logarithm for the Brownian motion, we deduce that there exists $t_2(\omega)$ such that

$$\forall t \leq t_2(\omega), \quad 0 \leq \frac{1}{t} e^{\sigma W_t(\omega) + \gamma t} \int_0^t e^{-\sigma W_u(\omega) - \gamma u} du \leq \frac{1}{t} e^{\sigma t^{\frac{1}{2}-\epsilon} + \gamma t} \int_0^t e^{\sigma u^{\frac{1}{2}-\epsilon} - \gamma u} du.$$

Denote $g(t) = \frac{1}{t} e^{\sigma t^{\frac{1}{2}-\epsilon} + \gamma t} \int_0^t e^{\sigma u^{\frac{1}{2}-\epsilon} - \gamma u} du$ and let us investigate the order in time near zero of this function. We have that

$$\begin{aligned} e^{\sigma t^{\frac{1}{2}-\epsilon} + \gamma t} &= 1 + \sigma t^{\frac{1}{2}-\epsilon} + \mathcal{O}(t^{1-2\epsilon}) \\ \int_0^t e^{\sigma u^{\frac{1}{2}-\epsilon} - \gamma u} du &= t + \frac{\sigma}{\frac{3}{2}-\epsilon} t^{\frac{3}{2}-\epsilon} + \mathcal{O}(t^{2-2\epsilon}) \end{aligned}$$

hence

$$g(t) = 1 + (\sigma + \frac{\sigma}{\frac{3}{2}-\epsilon}) t^{\frac{1}{2}-\epsilon} + \mathcal{O}(t^{1-2\epsilon}),$$

so $X_t(\omega) \leq \log(g(t)) \underset{t\to 0}{\sim} (\sigma + \frac{\sigma}{\frac{3}{2}-\epsilon}) t^{\frac{1}{2}-\epsilon}$, which ends the proof for X_t. □

Proposition 12 — *$(L_t)_{t\in[0,T]}$ is a martingale and, consequently, for all $g : \mathcal{C}([0,T]) \to \mathbb{R}$ measurable, the random variables $g((X_t)_{0\leq t\leq T})$ and $g((Z_t)_{0\leq t\leq T}) L_T$ are simultaneously integrable and then*

$$\mathbb{E}\Big(g((X_t)_{0\leq t\leq T})\Big) = \mathbb{E}\Big(g((Z_t)_{0\leq t\leq T}) L_T\Big).$$

Proof : The proof is similar to the proof of Proposition 4.

We have already shown existence and strong uniqueness for both SDE (16) and (17). Showing that the stopping time

$$\tau_n(Y) = \inf\left\{ t \in \mathbb{R}^+ \text{ such that } \int_0^t \left(\frac{e^{-Y_s} - 1 + Y_s}{\sigma s}\right)^2 ds \geq n \right\}, \text{ with the convention } \inf\{\emptyset\} = +\infty,$$

have infinite limits when n tends to $+\infty$, $\mathbb{Q}_X$ and $\mathbb{Q}_Z$ almost surely, follows from the previous lemma. □

One has

$$L_T = \exp\left[\int_0^T \frac{e^{-Z_t} - 1 + Z_t}{\sigma^2 t} dZ_t - \int_0^T \frac{e^{-Z_t} - 1 + Z_t}{\sigma^2 t} \left(\frac{e^{-Z_t} - 1 + Z_t}{2t} + \gamma - \frac{Z_t}{t}\right) dt\right].$$

Set $A(t,z) = \dfrac{1 - z + \frac{z^2}{2} - e^{-z}}{\sigma^2 t}$. The function $A :]0,T] \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable in time and twice continuously differentiable in space. So, we can apply Itô's Lemma on the interval $[\epsilon, T]$ for $\epsilon > 0$:

$$A(T, Z_T) = A(\epsilon, Z_\epsilon) + \int_\epsilon^T \frac{e^{-Z_t} - 1 + Z_t}{\sigma^2 t} dZ_t - \int_\epsilon^T \frac{1 - Z_t + \frac{Z_t^2}{2} - e^{-Z_t}}{\sigma^2 t^2} dt + \int_\epsilon^T \frac{1 - e^{-Z_t}}{2t} dt$$

Using the lemma 9, we let $\epsilon \to 0$ to obtain

$$A(T, Z_T) = \int_0^T \frac{e^{-Z_t} - 1 + Z_t}{\sigma^2 t} dZ_t - \int_0^T \frac{1 - Z_t + \frac{Z_t^2}{2} - e^{-Z_t}}{\sigma^2 t^2} dt + \int_0^T \frac{1 - e^{-Z_t}}{2t} dt.$$

Then

$$L_T = \exp\left[A(T, Z_T) - \int_0^T \phi(t, Z_t) dt\right]$$

where ϕ is the mapping

$$\phi(t,z) = \frac{e^{-z} - 1 + z - \frac{z^2}{2}}{\sigma^2 t^2} + \frac{1 - e^{-z}}{2t} + \frac{e^{-z} - 1 + z}{\sigma^2 t}\left(\frac{e^{-z} - 1 + z}{2t} + \gamma - \frac{z}{t}\right). \tag{19}$$

By (15) and Proposition 12, we get

$$C_0 = \mathbb{E}\left(e^{-rT} f(S_0 e^{Z_T}) \exp\left[A(T, Z_T) - \int_0^T \phi(t, Z_t) dt\right]\right). \tag{20}$$

Since for each $t > 0$, $\lim\limits_{z \to -\infty} \phi(t,z) = +\infty$ and $\lim\limits_{z \to +\infty} \phi(t,z) = -\infty$, it is not possible to apply the exact algorithm. One can use the unbiased estimator, at least theoretically, if there exists a random variable c_Z measurable with respect to Z such that

$$\mathbb{E}\left(e^{A(T,Z_T) - (r + c_Z)T} |f(S_0 e^{Z_T})| e^{\int_0^T |c_Z - \phi(t, Z_t)| dt}\right) < \infty.$$

Unfortunately, this reinforced integrability condition is never satisfied :

Lemma 13 — *Assume that f is a non identically zero function. Let p_Z and q_Z denote respectively a positive probability measure on $\mathbb{N}$ and a positive probability density on $[0,T]$. Let N be distributed according to p_Z and $(U_i)_{i \in \mathbb{N}^*}$ be a sequence of independent random variables identically distributed according to the density q_Z, both independent conditionally on the process $(Z_t)_{t \in [0,T]}$. Then the random variable*

$$e^{A(T,Z_T) - rT} f(S_0 e^{Z_T}) \frac{1}{p_Z(N)\, N!} \prod_{i=1}^N \frac{-\phi(U_i, Z_{U_i})}{q_Z(U_i)} \tag{21}$$

is non integrable.

Proof : By conditioning on Z, one has

$$\begin{aligned} \Delta := \mathbb{E}\left(\frac{e^{A(T,Z_T) - rT} |f(S_0 e^{Z_T})|}{p_Z(N)\, N!} \prod_{i=1}^N \frac{|\phi(U_i, Z_{U_i})|}{q_Z(U_i)}\right) &= \mathbb{E}\left(e^{A(T,Z_T) - rT} |f(S_0 e^{Z_T})| e^{\int_0^T |\phi(t, Z_t)| dt}\right) \\ &\geq \mathbb{E}\left(e^{A(T,Z_T) - rT} |f(S_0 e^{Z_T})| e^{\int_{\frac{T}{2}}^T |\phi(t, Z_t)| dt}\right) \end{aligned}$$

One can easily show that, $\forall z < 0$ and $\forall t \in [\frac{T}{2}, T]$, $\phi(t,z) \geq \overline{\phi}(z)$ where

$$\overline{\phi}(z) = \frac{e^{-z} - 1 + z - \frac{z^2}{2}}{\sigma^2(\frac{T}{2})^2} + \frac{e^{-z} - 1 + z}{\sigma^2 \frac{T}{2}} \left(\frac{e^{-z} - 1 + z}{T} + \gamma^+ - 2\frac{z}{T} \right)$$

Since $\overline{\phi}(z) \underset{-\infty}{\sim} 2\frac{e^{-2z}}{\sigma^2 T^2}$, there exists $c < 0$ such that for all $z < c$, $\overline{\phi}(z) \geq \frac{e^{-2z}}{\sigma^2 T^2}$. Hence,

$$\begin{aligned}
\Delta &\geq \mathbb{E}\left(e^{A(T,Z_T) - rT} |f(S_0 e^{Z_T})| e^{\frac{1}{\sigma^2 T^2} \int_{\frac{T}{2}}^{T} e^{-2Z_t} \mathbb{1}_{\{Z_t < c\}} dt} \right) \\
&\geq \mathbb{E}\left(e^{A(T,Z_T) - rT} |f(S_0 e^{Z_T})| e^{-\frac{e^{-2c}}{2\sigma^2 T}} e^{\frac{1}{\sigma^2 T^2} \int_{\frac{T}{2}}^{T} e^{-2Z_t} dt} \right)
\end{aligned}$$

Using Jensen's inequality we get

$$\Delta \geq \mathbb{E}\left(e^{A(T,Z_T) - rT} |f(S_0 e^{Z_T})| e^{-\frac{e^{-2c}}{2\sigma^2 T}} \exp\left(\frac{1}{2\sigma^2 T} e^{-\frac{4}{T} \int_{\frac{T}{2}}^{T} Z_t dt} \right) \right)$$

We have seen in the proof of lemma 11 that $Z_t = \frac{\sigma}{t} B_{\frac{t^3}{3}} + \frac{\gamma}{2} t$ where $(B_t)_{t \geq 0}$ is a standard Brownian motion. So, conditionally on Z_T, $\int_{\frac{T}{2}}^{T} Z_t dt$ is a gaussian random variable and hence $\Delta = +\infty$.

□

We are in a situation where $e^{A(T,Z_T)-rT}|f(S_0 e^{Z_T})| \mathbb{E}\left[\left| \frac{1}{p_Z(N)\, N!} \prod_{i=1}^{N} \frac{-\phi(U_i, Z_{U_i})}{q_Z(U_i)} \right| \Big| (Z_t)_{t \in [0,T]} \right]$ is non integrable while $e^{A(T,Z_T)-rT}|f(S_0 e^{Z_T})| \left| \mathbb{E}\left[\frac{1}{p_Z(N)\, N!} \prod_{i=1}^{N} \frac{-\phi(U_i, Z_{U_i})}{q_Z(U_i)} \Big| (Z_t)_{t \in [0,T]} \right] \right|$ is integrable since $\mathbb{E}\left(e^{-rT} |f(S_0 e^{Z_T})| \exp\left[A(T,Z_T) - \int_0^T \phi(t, Z_t) dt \right] \right) < \infty$. Then, a natural idea would consist in considering, for a given $n \in \mathbb{N}^*$, the random variable

$$e^{A(T,Z_T)-rT}|f(S_0 e^{Z_T})| \mathbb{E}\left[\left| \frac{1}{n} \sum_{j=1}^{n} \frac{1}{p_Z(N_j)\, N_j!} \prod_{i=1}^{N_j} \frac{-\phi(U_i^j, Z_{U_i^j})}{q_Z(U_i^j)} \right| \Big| (Z_t)_{t \in [0,T]} \right]$$

where $(N_j)_{1 \leq j \leq n}$ are independent variables having the same law as N and $\left((U_i^j)_{i \in \mathbb{N}^*} \right)_{1 \leq j \leq n}$ are independent sequences having the same law as $(U_i)_{i \in \mathbb{N}^*}$, both independent conditionally on the process $(Z_t)_{t \in [0,T]}$. The following general result tells us that this is not sufficient to circumvent integrability problems.

Lemma 14 — *Let Y and Z be two real random variables and $g : \mathbb{R} \to \mathbb{R}$ a given measurable function. Assume that $g(Z)\mathbb{E}(Y|Z)$ is integrable while $g(Z)\mathbb{E}(|Y|\,|Z)$ is non integrable. Then, when $(Y_i)_{1 \leq i \leq n}$ is a sequence of independent random variables having the same law as Y, $\forall n \in \mathbb{N}^*$, the random variable $g(Z)\mathbb{E}\left(|\frac{1}{n}\sum_{i=1}^{n} Y_i|\, |Z \right)$ is non integrable.*

Proof : Denote by e, e_1 and e_n three functions satisfying

$$\forall z \in \mathbb{R}, \quad e(z) = \mathbb{E}(Y|Z=z), \quad e_1(z) = \mathbb{E}(|Y_1|\,|Z=z) \quad \text{and } e_n(z) = \mathbb{E}\left(\left| \frac{1}{n} \sum_{i=1}^{n} Y_i \right| |Z = z \right)$$

On the one hand, since $\int_{\mathbb{R}} |g(z)|\,|e(z)| \mathbb{P}_Z(dz) < \infty$ and $\int_{\mathbb{R}} |g(z)|\, e_1(z) \mathbb{P}_Z(dz) = +\infty$, where $\mathbb{P}_Z$ is the law of Z, we have that $\int_{\mathbb{R}} |g(z)|\, e_1(z) \mathbb{1}_{\{e_1(z) \geq 2|e(z)|\}} \mathbb{P}_Z(dz) = +\infty$.

On the other hand, $\forall z \in \mathbb{R}$,

$$\begin{aligned}
e_n(z) &\geq \frac{1}{n}\left[\mathbb{E}\left(\left|\sum_{i=1}^n Y_i\right| \mathbb{1}_{\{\forall 2\leq j\leq n, Y_j\geq 0\}}|Z=z\right) + \mathbb{E}\left(\left|\sum_{i=1}^n Y_i\right| \mathbb{1}_{\{\forall 2\leq j\leq n, Y_j< 0\}}|Z=z\right)\right]\\
&\geq \tfrac{1}{n}\left[\mathbb{E}\left(Y_1^+|Z=z\right)\mathbb{P}\left(Y_1\geq 0|Z=z\right)^{n-1} + \mathbb{E}\left(Y_1^-|Z=z\right)\mathbb{P}\left(Y_1< 0|Z=z\right)^{n-1}\right]\\
&= \tfrac{1}{n}\left[\tfrac{e_1(z)+e(z)}{2}\mathbb{P}\left(Y_1\geq 0|Z=z\right)^{n-1} + \tfrac{e_1(z)-e(z)}{2}\mathbb{P}\left(Y_1< 0|Z=z\right)^{n-1}\right]\\
&\geq \tfrac{1}{n}\left[\tfrac{e_1(z)}{4}\mathbb{1}_{\{e_1(z)\geq 2|e(z)|\}}\mathbb{P}\left(Y_1\geq 0|Z=z\right)^{n-1} + \tfrac{e_1(z)}{4}\mathbb{1}_{\{e_1(z)\geq 2|e(z)|\}}\mathbb{P}\left(Y_1< 0|Z=z\right)^{n-1}\right]\\
&\geq \tfrac{e_1(z)}{n2^n}\mathbb{1}_{\{e_1(z)\geq 2|e(z)|\}}
\end{aligned}$$

Hence, $\mathbb{E}\left[g(Z)\mathbb{E}\left(\left|\frac{1}{n}\sum_{i=1}^n Y_i\right| |Z\right)\right] = \int_{\mathbb{R}} |g(z)|e_n(z)\mathbb{P}_Z(dz) = +\infty$. □

There is still hope yet. In the proof of Lemma 13, we saw that integrability problems appear when Z_t takes large negative values so that $\phi(t, Z_t)$ tends rapidly towards $+\infty$. Since $\lim_{z\to+\infty}\phi(t,z) = -\infty$, one possible issue is to split the function $\phi(t, Z_t)$ into a positive part and a negative part. The first term can be handled by the exact simulation technique whereas the second term, which as we shall see in the following section presents no integrability problems, can be handled by the unbiased estimator technique.

2.2.1 An hybrid pseudo-exact method

We rewrite (20) in the following form

$$C_0 = \mathbb{E}\left(e^{A(T,Z_T)-rT}f(S_0e^{Z_T})e^{\int_0^T \phi^-(t,Z_t)dt}e^{-\int_0^T \phi^+(t,Z_t)dt}\right). \tag{22}$$

Let p_Z and q_Z denote respectively a positive probability measure on $\mathbb{N}$ and a positive probability density on $[0, T]$. Let N be distributed according to p_Z and $(U_i)_{i\in\mathbb{N}^*}$ be a sequence of independent random variables identically distributed according to the density q_Z, both independent conditionally on the process $(Z_t)_{t\in[0,T]}$. Note that, since $e^{A(T,Z_T)-rT}f(S_0e^{Z_T})e^{\int_0^T|\phi^-(t,Z_t)|dt}e^{-\int_0^T \phi^+(t,Z_t)dt} = e^{A(T,Z_T)-rT}f(S_0e^{Z_T})e^{-\int_0^T \phi(t,Z_t)dt}$ is integrable, one has

$$C_0 = \mathbb{E}\left(e^{A(T,Z_T)-rT}f(S_0e^{Z_T})\frac{1}{p_Z(N)N!}\left(\prod_{i=1}^N \frac{\phi^-(U_i, Z_{U_i})}{q_Z(U_i)}\right)e^{-\int_0^T \phi^+(t,Z_t)dt}\right). \tag{23}$$

Remark 15 — *There is no hope that this estimator is square integrable. Indeed, one can show as in Lemma 13 that* $\mathbb{E}\left(e^{\int_0^T(\phi^-(t,Z_t))^2dt}\right) = +\infty$ *since* $(\phi^-(t,z))^2$ *is of order* z^4 *for large positive* z.

The idea then is to apply the exact simulation technique to simulate an event with probability $e^{-\int_0^T \phi^+(t,Z_t)dt}$. Since for each $t > 0$, $\lim_{z\to-\infty}\phi^+(t,z) = +\infty$, one needs to bound from above $\phi^+(t,z)$, uniformly with respect to $t \in [0, T]$, for $z > c$ where $c < 0$ is a given constant. Thanks to the following lemma, it is possible to do so but only uniformly with respect to $t \in [\epsilon, T]$ for all $\epsilon > 0$:

Lemma 16 — *For all* $0 < t \leq T$,

$$\sup_{z\geq 0}\phi^+(t,z) \leq \frac{\gamma^2}{\sigma^2} + \frac{\gamma}{\sigma^2 t} + \frac{1}{t}\left(\frac{1}{2} - \frac{\gamma}{\sigma^2}\right)^+$$

and

$$\forall c<0,\ \sup_{z\in[c,0]}\phi^+(t,z)\le \frac{e^{-c}-1+c}{\sigma^2t^2}(1+\gamma^+t)+\frac{(e^{-c}-1)^2}{2\sigma^2t^2}-\frac{c^2}{\sigma^2t^2}.$$

Proof : Let $z>0$. It is useful to distinguish two cases according to the sign of γ :

1. $\underline{\gamma\ge 0}$
   We rewrite ϕ in the following form
   $$\phi(t,z)=\frac{e^{-z}-1+z-\frac{z^2}{2}}{\sigma^2t^2}+\frac{1-e^{-z}}{t}\left(\frac{1}{2}-\frac{\gamma}{\sigma^2}\right)+\frac{\gamma z}{\sigma^2 t}-\frac{z^2-(z\wedge 1)^2}{2\sigma^2t^2}+\frac{(e^{-z}-1)^2-(z\wedge 1)^2}{2\sigma^2t^2}$$
   First note that $\frac{e^{-z}-1+z-\frac{z^2}{2}}{\sigma^2t^2}\le 0$, $\frac{1-e^{-z}}{t}\left(\frac{1}{2}-\frac{\gamma}{\sigma^2}\right)\le\frac{1}{t}\left(\frac{1}{2}-\frac{\gamma}{\sigma^2}\right)^+$ and $\frac{(e^{-z}-1)^2-(z\wedge1)^2}{2\sigma^2t^2}\le 0$. Moreover,
   $$\begin{aligned}\frac{\gamma z}{\sigma^2 t}-\frac{z^2-(z\wedge 1)^2}{2\sigma^2t^2} &= \frac{1}{\sigma^2}\left(\gamma\frac{z}{t}-\frac{1}{2}\left(\frac{z}{t}\right)^2+\frac{(\frac{z}{t}\wedge\frac{1}{t})^2}{2}\right)\\ &\le \begin{cases}\frac{\gamma}{\sigma^2 t} & \text{if } \gamma t\le 1\\ \frac{\gamma^2}{\sigma^2} & \text{otherwise}\end{cases}\end{aligned}$$
   Consequently, $\phi^+(t,z)\le\frac{\gamma^2}{\sigma^2}+\frac{\gamma}{\sigma^2t}+\frac{1}{t}\left(\frac{1}{2}-\frac{\gamma}{\sigma^2}\right)^+$.
2. $\underline{\gamma\le 0}$
   Now we rewrite ϕ in the following form
   $$\phi(t,z)=\frac{e^{-z}-1+z-\frac{z^2}{2}}{\sigma^2t^2}+\gamma\frac{e^{-z}-1+z}{\sigma^2t}+\frac{(e^{-z}-1)^2-z^2}{2\sigma^2t^2}+\frac{1-e^{-z}}{2t}$$
   It is then easy to show that $\phi^+(t,z)\le\frac{1}{2t}$.

Note that $\frac{1}{2t}\le\frac{\gamma^2}{\sigma^2}+\frac{\gamma}{\sigma^2t}+\frac{1}{t}\left(\frac{1}{2}-\frac{\gamma}{\sigma^2}\right)^+$. Hence, gathering the two cases yields the first part of the lemma. Let now $z\in[c,0]$ for a given negative constant c. We rewrite ϕ in the following form

$$\phi(t,z)=\underbrace{\frac{e^{-z}-1+z}{\sigma^2t^2}(1+\gamma^+t)+\frac{(e^{-z}-1)^2}{2\sigma^2t^2}-\frac{z^2}{\sigma^2t^2}}_{\ge 0\text{ for }z<0}+\underbrace{\frac{1-e^{-z}}{2t}-\gamma^-\frac{e^{-z}-1+z}{\sigma^2t}}_{\le 0\text{ for }z<0}.$$

Since $\partial_z\left[\frac{e^{-z}-1+z}{\sigma^2t^2}(1+\gamma^+t)+\frac{(e^{-z}-1)^2}{2\sigma^2t^2}-\frac{z^2}{\sigma^2t^2}\right]=\frac{1-e^{-2z}-2z+t\gamma^+(1-e^{-z})}{t^2\sigma^2}$ is negative for all $z<0$, one has that

$$\sup_{z\in[c,0]}\phi^+(t,z)\le\frac{e^{-c}-1+c}{\sigma^2t^2}(1+\gamma^+t)+\frac{(e^{-c}-1)^2}{2\sigma^2t^2}-\frac{c^2}{\sigma^2t^2}.$$

□

This lemma suggests to apply the exact algorithm on $[\epsilon,T]$ for a fixed positive threshold ϵ. It remains to handle the time interval $[0,\epsilon[$. Thanks to the following lemma, we that $\phi^+(t,Z_t)$ can be approximately bounded from above for small t, almost surely, by a function of t. The idea is then to extend the exact simulation algorithm by simulating an inhomogeneous Poisson process. Of course, this hybrid method is no longer exact since the positive threshold for which the upper bound holds is random.

Lemma 17 — *For all $\eta > 0$, there exists a random neighborhood of $t = 0$ such that*

$$\phi^+(t, Z_t) \le \left(\frac{2c^3}{3\sigma^2} + \frac{c}{2}\right) t^{-\frac{1}{2}-\eta} \tag{24}$$

where $c = \max(\frac{\sigma}{3^{\frac{1}{2}-\frac{\eta}{3}}}, \frac{\gamma}{2})$.

Proof : We rewrite (19) this way

$$\phi(t,z) = \left(\frac{1-e^{-z}}{2} + \gamma\frac{e^{-z}-1+z}{\sigma^2}\right)\frac{1}{t} - \left(\frac{1-z+\frac{z^2}{2}-e^{-z}-\frac{1}{2}(e^{-z}-1+z)(e^{-z}-1-z)}{\sigma^2}\right)\frac{1}{t^2}$$

and make the following Taylor expansions

$$\frac{1-z+\frac{z^2}{2}-e^{-z}-\frac{1}{2}(e^{-z}-1+z)(e^{-z}-1-z)}{\sigma^2} = \frac{2}{3\sigma^2}z^3 + \mathcal{O}(z^4)$$

$$\text{and } \frac{1-e^{-z}}{2} + \gamma\frac{e^{-z}-1+z}{\sigma^2} = \frac{1}{2}z + \mathcal{O}(z^2).$$

On the other hand, we have seen in the proof of lemma 11 that there exists a random neighborhood of zero such that $Z_t \le ct^{\frac{1}{2}-\eta}$ where $c = \max(\frac{\sigma}{3^{\frac{1}{2}-\frac{\eta}{3}}}, \frac{\gamma}{2})$. We conclude that, in a random neighborhood of zero,

$$\phi^+(t, Z_t) \le \left(\frac{2c^3}{3\sigma^2} + \frac{c}{2}\right) t^{-\frac{1}{2}-\eta}.$$

□

2.2.2 Numerical computation

For numerical computation, we are going to use the following set of parameters : $S_0 = 100$, $K = 100$, $\sigma = 0.2$, $r = 0.1$, $\delta = 0$ and $T = 1$. To fix the ideas, let us consider a call option. The price C_0 writes as follows

$$C_0 = \mathbb{E}\left(e^{A(T,Z_T)-rT}\left(S_0e^{Z_T}-K\right)^+\left(e^{c_p}\prod_{i=1}^{N}\frac{\phi^-(U_i, Z_{U_i})}{c_p}\right)e^{-\int_0^T \phi^+(t,Z_t)dt}\right).$$

where $N \sim \mathcal{P}(c_p)$ and $(U_i)_{i\ge1}$ is an independent sequence of independent random variables uniformly distributed in $[0,T]$. The parameter $c_p > 0$ is set to one in the following. We give a description of the hybrid method we implement :

Algorithm 2
On the time interval $I_j := [\frac{T}{2^{j+1}}, \frac{T}{2^j}]$,

1. *Simulate* $Z_{\frac{T}{2^{j+1}}}, Z_{\frac{T}{2^j}}$ *and a lower bound* m_j *for the minimum of* $(Z_t)_{t\in I_j}$ *(use the fact that* $Z_t = \frac{\sigma}{t}B_{\frac{t^3}{3}} + \frac{\gamma}{2}t$ *where* $(B_t)_{t\ge0}$ *is a standard Brownian motion).*
2. *Find* $M^j > 0$ *such that* $\forall t \in I_j, \phi^+(t, Z_t) \le M^j$ *(use Lemma 16).*
3. *Simulate an homogeneous spatial Poisson process on the rectangle* $I_j \times [0, M^j]$ *and accept (respectively reject) the trajectory simulated if the number of points falling below the graph* $(\phi^+(t, Z_t))_{t\in I_j}$ *is equal to (respectively different from) zero.*

Carry on this acceptance rejection algorithm until reaching a time interval I_J for a chosen $J \in \mathbb{N}^*$. On the remaining time interval $[0, \frac{T}{2^{J+1}}]$, use the same acceptance/rejection algorithm but with an inhomogeneous spatial Poisson process this time (use Lemma 17).

In table 3, we give the price obtained by our method for different values of the positive threshold $\epsilon = \frac{T}{2^{J+1}}$. The number M of Monte Carlo simulations is equal to 10^5 and the true price is equal to 7.042 (computed using a Monte Carlo method with a trapezoidal scheme and a Kemna-Vorst control variate technique).

	Price	CPU
$\epsilon = \frac{T}{2^2}$	6.9394	7s
$\epsilon = \frac{T}{2^4}$	6.9590	10s
$\epsilon = \frac{T}{2^6}$	6.9703	13s
$\epsilon = \frac{T}{2^8}$	6.9952	17s
$\epsilon = \frac{T}{2^{10}}$	7.0423	21s

Table 3: Price of the Asian call using the hybrid-pseudo exact method.

Clearly, the method is not yet competitive regarding computation time. Nevertheless, unlike the usual discretization methods, it is not prone to discretization errors.

3 Conclusion

In this article, we have applied two original Monte Carlo methods for pricing Asian like options which have the following pay-off : $(\alpha S_T + \beta \int_0^T S_t dt - K)_+$. In the case $\alpha \neq 0$, we applied both the algorithm of Beskos *et al.* [1] and a method based on the unbiased estimator of Wagner [23] and more recently the Poisson estimator of Beskos *et al.* [2] and the generalized Poisson estimator of Fearnhead *et al.* [6]. The numerical results show that the latter performs the best. The more interesting case $\alpha = 0$, which corresponds to usual continuously monitored Asian options, can not be treated using neither the exact algorithm, nor the method of exact computation of expectation but we investigate an hybrid pseudo-exact method which combines the two techniques. More generally, this hybrid method is an extension of the two exact methods and can be applied in other situations.

From a practical point of view, the main contribution of these techniques is to allow Monte Carlo pricing without resorting to discretization schemes. Hence, we are no longer prone to the discretization bias that we encounter in standard Monte Carlo methods for pricing Asian like options. Even though these exact methods are time consuming, they provide a good and reliable benchmark.

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4 Appendix

4.1 The practical choice of p and q in the U.E method

The best choice for the probability law p of N and the common density q of the variables $(V_i)_{i\geq 1}$ is obviously the one for which the variance of the simulation is minimum. In a very general setting, it is difficult to tackle this issue. In order to have a first idea, we are going to restrict ourselves to the computation of $\mathbb{E}\left(\frac{1}{p(N)\,N!}\prod_{i=1}^{N}\frac{g(V_i)}{q(V_i)}\right)$ where $g:[0,T]\to\mathbb{R}$.

Lemma 18 — *When g is a measurable function on $[0,T]$ such that $0<\int_0^T|g(t)|dt<+\infty$, the variance of $\frac{1}{p(N)\,N!}\prod_{i=1}^{N}\frac{g(V_i)}{q(V_i)}$ is minimal for*

$$q_{opt}(t)=\frac{|g(t)|}{\int_0^T|g(t)|dt}\,\mathbb{1}_{[0,T]}(t)\ \ \textit{and}\ \ p_{opt}(n)=\frac{\left(\int_0^T|g(t)|dt\right)^n}{n!}\exp\left(-\int_0^T|g(t)|dt\right).$$

Proof : Minimizing the variance in (7) comes down to minimizing the expectation of the square of $\frac{1}{p(N)\,N!}\prod_{i=1}^{N}\frac{g(V_i)}{q(V_i)}$.
Set

$$F(p,q)=\mathbb{E}\left(\frac{1}{(p(N)\,N!)^2}\prod_{i=1}^{N}\frac{g^2(V_i)}{q^2(V_i)}\right)=\sum_{n=0}^{+\infty}\frac{\left(\int_0^T\frac{g^2(t)}{q(t)}dt\right)^n}{p(n)\,(n!)^2}.$$

Using Cauchy-Schwartz inequality we obtain a lower bound for $F(p,q)$

$$\begin{aligned}
F(p,q)=\sum_{n=0}^{+\infty}\left(\frac{\left(\int_0^T\frac{g^2(t)}{q(t)}dt\right)^{\frac{n}{2}}}{p(n)\,n!}\right)^2 p(n) &\geq \left(\sum_{n=0}^{+\infty}\frac{\left(\int_0^T\frac{g^2(t)}{q(t)}dt\right)^{\frac{n}{2}}}{n!}\right)^2\\
&= \left(\sum_{n=0}^{+\infty}\frac{\left(\int_0^T\left(\frac{g(t)}{q(t)}\right)^2q(t)dt\right)^{\frac{n}{2}}}{n!}\right)^2\\
&\geq \left(\sum_{n=0}^{+\infty}\frac{\left(\int_0^T|g(t)|dt\right)^n}{n!}\right)^2\\
&= \exp\left(2\int_0^T|g(t)|dt\right).
\end{aligned}$$

We easily check that this lower bound is attained for q_{opt} and p_{opt}.

□

The optimal probability distribution p_{opt} is the Poisson law with parameter $\int_0^T|g(t)|dt$. This justifies our use of a Poisson distribution for p.

4.2 Simulation from the distribution h given by (13)

Recall that

$$h(u) = C\exp\left(A(u) - \frac{(u-X_0)^2}{2T}\right) = C\exp\left(\frac{\gamma}{\sigma}u + \frac{\beta S_0}{\sigma^2}(1-e^{-\sigma u}) - \frac{(u-X_0)^2}{2T}\right)$$

where C is a normalizing constant.
The expansion of the exponential $e^{-\sigma u}$ at the first order yields

$$h(u) \approx C\exp\left(\frac{\gamma}{\sigma}u + \frac{\beta S_0}{\sigma}u - \frac{(u-X_0)^2}{2T}\right) = C\exp\left(-\frac{(u-(X_0+\frac{T(\gamma+\beta S_0)}{\sigma}))^2}{2T}\right).$$

This suggests to do rejection sampling using the normal distribution with mean $X_0 + \frac{T(\gamma+\beta S_0)}{\sigma}$ and variance T as prior. Unfortunately, for a standard set of parameters, this method gives bad results. Even a second order expansion of $e^{-\sigma u}$ which also modifies the variance does not work.

In order to get round this problem, we evaluate the mode u^* of h. We have

$$h'(u^*) = C\left(\frac{\gamma}{\sigma} + \frac{\beta S_0}{\sigma}e^{-\sigma u^*} - \frac{u^*-X_0}{T}\right)\exp\left(\frac{\gamma}{\sigma}u^* + \frac{\beta S_0}{\sigma^2}(1-e^{-\sigma u^*}) - \frac{(u^*-X_0)^2}{2T}\right).$$

So, $h'(u^*) = 0$ if and only if

$$\frac{\gamma}{\sigma} + \frac{\beta S_0}{\sigma}e^{-\sigma u^*} - \frac{u^*-X_0}{T} = 0$$

which writes

$$\sigma(u^* - X_0 - \frac{\gamma}{\sigma}T)e^{\sigma(u^*-X_0-\frac{\gamma}{\sigma}T)} = T\beta S_0 e^{-\sigma X_0 - \gamma T}.$$

The function $x \mapsto xe^x$ is continuous and increasing on $[0,+\infty[$ and so is its inverse which we denote by W. Since $T\beta S_0 e^{-\sigma X_0-\gamma T} \geq 0$, we deduce that h is unimodal and that its mode satisfies

$$u^* = \frac{\gamma T + W\left(\beta S_0 T e^{-\gamma T - \sigma X_0}\right) + \sigma X_0}{\sigma}.$$

The function W is the well-known Lambert function, also called the Omega function. It is uniquely valued on $[0,+\infty[$ and there are robust and fast numerical methods based on series expansion for approximating this function (see for example Corless *et al.* [4]).

Numerical tests showed that performing rejection sampling using a Gaussian distribution with variance T and mean u^* instead of $X_0 + \frac{T(\gamma+\beta S_0)}{\sigma}$ gives plain satisfaction. In table 4, we see that for arbitrary choice of the parameter $\frac{\alpha}{\alpha+\beta}$, the acceptance rate of the algorithm is always high (of order 70%) and that the computation time is low.

$\frac{\alpha}{\alpha+\beta}$	Nb of simulations	Acceptance rate	Computation time
0.2		61%	3s
0.5	10^6	68%	3s
0.8		80%	2s

Table 4: Acceptance rate of the rejection algorithm of simulating from the distribution h in (13) with $S_0 = 100, \sigma = 0.3, T = 2$ and $r = 0.1$.